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Digital Phase and Amplitude Imbalance Correction in Analog Quadrature Signals

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Introduction

Quadrature, or complex-valued, signals refer to waveforms that are deemed "analytic". In this context, an analytic signal is one which has a frequency response containing only positive frequency terms. In this paper, the terms "quadrature" and "analytic" are sometimes used interchangeably.

Analog quadrature hybrids are commonplace in creating a quadrature analog signal. These are popular devices for phase detectors, phase noise measurement systems, and analog quadrature mixers. One issue with analog hybrids is there linearity over frequency and temperature. Most hybrids are hand tuned to give the best performance over their specified bandwidth. Even so, the hybrid will have inherent amplitude and phase imbalances over frequency. These imbalances cause unwanted distortion in the output of the system. For systems that require stringent quadrature specs over wide operating temperature and frequency, these limitations can yield poor performance.

One way to mitigate these imbalances is to digitally correct for the phase and amplitude imbalances before further processing. In today's systems, most analog signals are digitized and processed in the digital domain. So somewhere in the signal chain we will be able to apply digital signal processing methods to rectify analog deficiencies.

Quadrature Signals Redux

Recall from Fourier theory, all real-valued signals have "symmetric" positive and negative frequency terms. For analytic signals, the negative portion of the frequency response disappears. Analytic signals are also related to Hilbert transforms, as they can be represented by:

$$x_{quad}(t) = x(t) + j \cdot H\{x(t)\} \qquad (1.1)$$

Where H{x(t)}is the Hilbert transform of the real signal x(t). By definition, the Hilbert transform shifts the input signal 90 degrees for negative frequencies, and -90 degrees for positive frequencies [2]. If x(t) is a sinusoid, you can easily see that the Hilbert transform turns sines into cosines, and vice versa. As such, passing a real-valued signal through a perfect Hilbert transform creates a phase-shifted version of the input. Combining both real-valued signals into a single complex-valued waveform creates an "analytic" (or quadrature) signal. Analog hybrids perform this Hilbert transform via a 90 degree phase shifter.

As an example, let $x(t) = \cos(2\pi f_o t)$. This real-valued signal has a symmetric Fourier transform: $X(f) = \frac{1}{2}[\delta(f - f_o) + \delta(f + f_o)]$. As stated before, X(f) has a positive and negative frequency term, and this is true for any real-valued signal [6].

The Hilbert transform of x(t) is:

$$H\{x(t)\} = \cos(2\pi f_o t - \frac{\pi}{2}),$$

which is equal to $\sin(2\pi f_o t)$. So, our quadrature signal takes on the form:

$$x_{quad}(t) = \cos(2\pi f_0 t) + j\sin(2\pi f_o t).$$

By using Euler's identity $e^{jx} = \cos(x) + j\sin(x)$, we get:

$$x_{quad}(t) = e^{j2\pi f_o t}$$

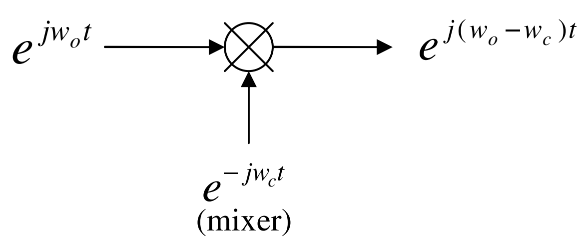
The result is a complex-valued exponential. Recall the Fourier transform of a complex exponential is: $X_{quad}(f) = \delta(f - f_o)$. Hence, our quadrature signal now only has a positive frequency term, the negative frequency term has conveniently "vanished". More on this concept will be presented next.

Quadrature mixing

A common signal processing task is to relocate signals to other frequencies where we can easily process them. Baseband signal processing is the cornerstone of many communication schemes and other DSP algorithms. Here we are attempting to mix the input signals down near DC where we can process them more efficiently.

Quadrature mixing is the process of taking a complex or real-valued discretized input and mixing it with a complex-valued exponential. In this case, we have 2 data streams now, the I (in-phase) stream and the Q (quadrature phase) stream. Each signal stream, taken individually, is a real-valued signal. But when we combine them into a complex-valued signal, we will see some remarkable properties.

Assume we have a complex-valued exponential at frequency $w_o = 2\pi f_o$, and a complex exponential mixer at frequency $-w_c$. We can show that for a perfect input signal, we are mathematically mixing the quadrature signal to the frequency (w_o-w_c)



(1.1b)

Fig.1 – Complex mixer

The input is typically an analog quadrature signal that is further processed digitally. Thus we can also assume these signals have already been sampled. So the mixing operation is done after we have digitized each signal.

The mixer is typically implemented as a numerically-controlled oscillator (NCO), which is a fancy name for a type of lookup table that stores the mixer samples we wish to use. NCO analysis is not included in this paper, but suffice it to say that the NCO is simply a table of complex exponential values at frequency w_c.

A block diagram of a digital complex mixer is shown next.

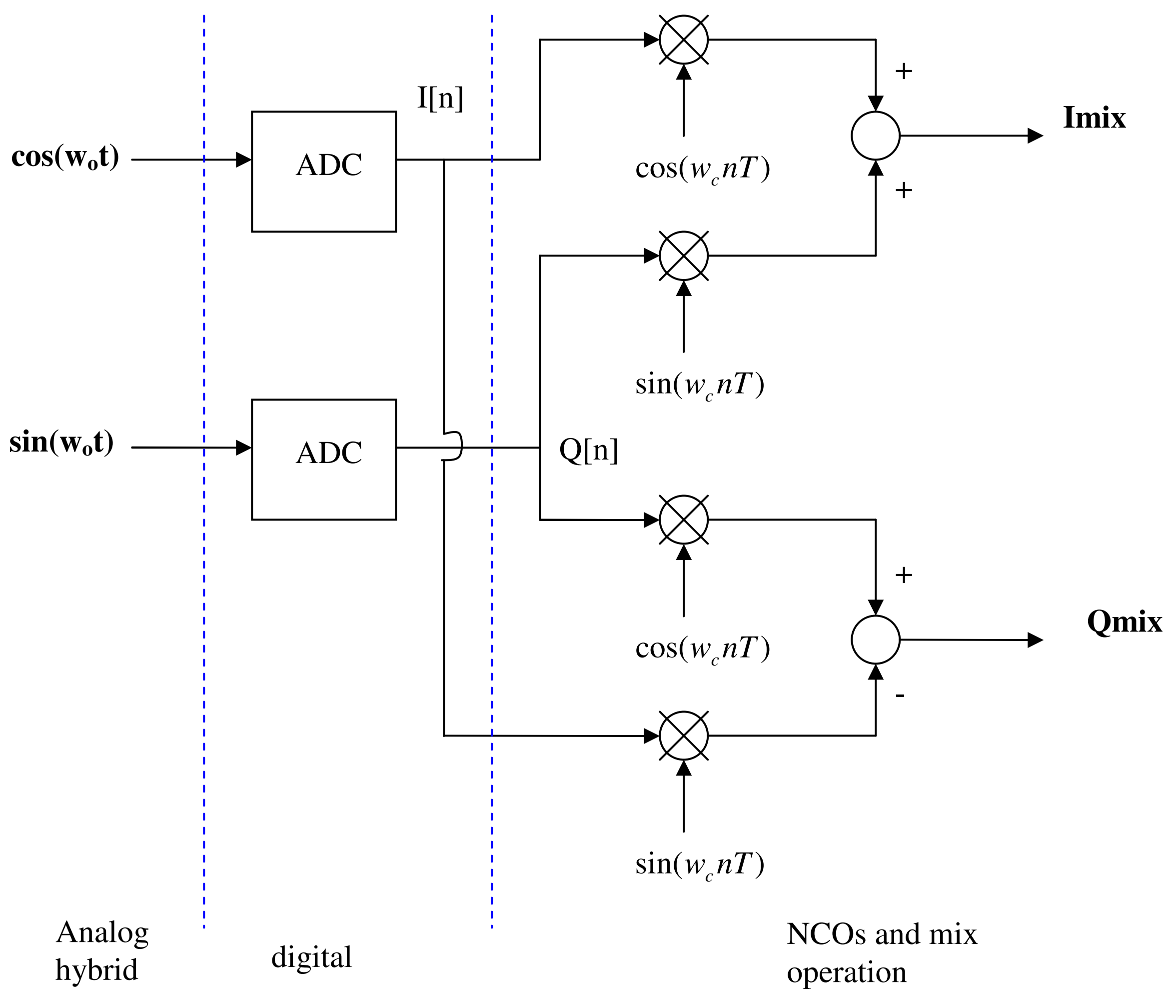


Fig. 2 – Block diagram of quadrature (complex) mixdown

In Fig.2 we have used the notation "nT" as the sampling function, where T = 1/Fs and n is an integer.

Fig. 2 may look complicated, but it is simply the expanded complex multiplication boiled down into real multiply and add operations. Again using Euler's identity:

$$e^{j(w_0-w_c)t} = [\cos(w_o nT) + j\sin(w_o nT)][\cos(w_c nT) - j\sin(w_c nT)] \quad (1.2)$$

Substituting $I[n] = \cos(w_o nT)$ and $Q[n] = \sin(w_o nT)$ for the input samples, we get:

$$e^{j(w_0-w_c)t} = \underbrace{I[n]\cos(w_c nT) + Q[n]\sin(w_c nT)}_{\text{Real part}} + j\{\underbrace{Q[n]\cos(w_c nT) - I[n]\sin(w_c nT)}_{\text{Imaginary part}}\}$$

And these are indeed the operations shown in Fig. 2.

$$I_{mix} = I[n]\cos(w_c nT) + Q[n]\sin(w_c nT) \quad \text{“real part”}$$
$$Q_{mix} = Q[n]\cos(w_c nT) - I[n]\sin(w_c nT) \quad \text{“imaginary part”}$$

It is interesting to note that you can switch between a down mix and up mix (with or without a phase shift) just by changing the sign of the additions in the above equations.

Phase and Amplitude Imbalance in Quadrature Signals

Another subject that is hardly discussed in DSP literature is the real-world implications of imperfect quadrature signaling. As we have seen, if we have perfect analytic signals, then the mathematics works out to our benefit. But what happens if the quadrature signal has amplitude differences between the I and Q channel? What happens if the I and Q channel are no longer 90 degrees out of phase relative to one another (no longer quadrature)?

In short, any amplitude or phase imbalance in the input quadrature signal will create distortion. Quadrature imbalances void the analytic signal definition and create a situation where system performance is limited to these errors. These sort of imperfections will have consequences in digital communications, demodulation routines, and phase angle detectors - just to name a few related DSP areas.

Phase imbalance

Let us first tackle phase imbalance. Here we start by assuming the analog quadrature signal has a phase error offset of φ radians. We can push this entire error into the Q term since we are worried about the relative phase between I and Q. We need the relative phase to be 90 degrees for a purely analytic signal, and any imbalance will show an error in the relative phase between I and Q.

In this case it can be shown that the imbalanced complex-valued signal takes on the form:

$$A_\varphi(t) = \cos(wt) + j\sin(wt + \phi) \quad (1.6)$$

And put into complex exponential notation Eq. (1.6) becomes:

$$A_{\phi}(t)=\frac{1}{2}\left[e^{jwt}\cdot(1+e^{j\phi})+e^{-jwt}\cdot(1-e^{-j\phi})\right] \quad (1.7)$$

Eq (1.7) shows that $A_{\varphi}(t)$ is no longer analytic, but is rather symmetric and complex-valued. It lies neither on the real or imaginary frequency axis, but somewhere inbetween. The spectrum of $A_{\varphi}(t)$ also has positive and negative frequency terms at *w* and *-w*, similar to a real-valued sinusoid.

NOTE: If we substitute φ=0, you can see that Eq. (1.7) turns back into its analytic e^{jwt}, which is what we expect with no phase imbalance.

Eq. (1.7) gives us a tool to assess what happens when the input signal to a quadrature mixer has a phase imbalance. It tells us that the phase imbalance creates a negative frequency image scaled by a complex number $\frac{1}{2}(1-e^{-j\phi})$. Also, the positive frequency is scaled - it decreases as the phase imbalance angle increases. Thus for increasing phase imbalance, the negative freq image gets stronger and the positive freq gets weaker. If φ=π, we can see that the negative freq image has unity amplitude, and the positive frequency image disappears. Since the phase angle offset directly affects the strength of the image frequency, it governs the amount of alias cancelation we will see when we attempt to combine the real signals into a complex-valued waveform.

Recall for an analytic signal, the absence of negative frequencies is due to the fact that they directly canceled during the analytic signal derivation. With a phase imbalance, now these negative frequency "images" will not completely cancel and will be directly proportional to the phase angle imbalance between the incoming analog signals.

If you reference Fig. 3c, remember when we multiplied the Q signal by "j", it effectively rotated the signal 90 degrees so it lined up with the real axis (since it was originally on the imaginary axis). As such, adding Fig. 3a with Fig. 3c resulted in a spectrum that resided on the real axis, and the negative (alias) images fully canceled. With a phase imbalance, you can imagine that the Q signal is no longer aligned on the imaginary axis, but it offset from it. So rotating this imbalanced Q signal 90 degrees does not align it on the real axis, but offset from it by φ degrees. When you add the real and imaginary signals, you have to do vector addition [5] on the I and Q spectral graphs, and you will see that the real part of the composite quadrature signal is no longer on the real axis. The result is that a portion of the negative (alias) frequency image remains due to incomplete cancelation.

The phase imbalanced quadrature signal has a frequency response given by:

$$I[n]=\cos(w_o nT) \Leftrightarrow I(f)=\frac{1}{2}[\delta(f-f_o)+\delta(f+f_o)]$$

$$Q[n]=\sin(w_o nT+\varphi) \Leftrightarrow Q(f)=-\frac{j}{2}e^{j\varphi}[\delta(f-f_o)-\delta(f+f_o)]$$

For the Q signal, we see the delta functions do not lie strictly on the "j" axis, but are now scaled by $e^{j\varphi}$, effectively moving them off the "j" axis.

Next is a plot showing what happens during the complex signal construction with a phase offset angle equal to φ. We use the same signal as in Fig. 3, where the real-valued I and Q signals are at $f_o > Fs/2$ (purposefully aliased). This can be done without loss of generality.

NOTE: If the I and Q signals do not violate the sampling theorem ($f_o < Fs/2$), then the same cancelation issue arises. As a formality, we don't refer to it as alias cancelation. We want to cancel the negative frequency images, which are not aliased in that case. But the concept is really the same - we are attempting to make the negative images in the real signals "disappear" through construction combination of the real-valued signals.

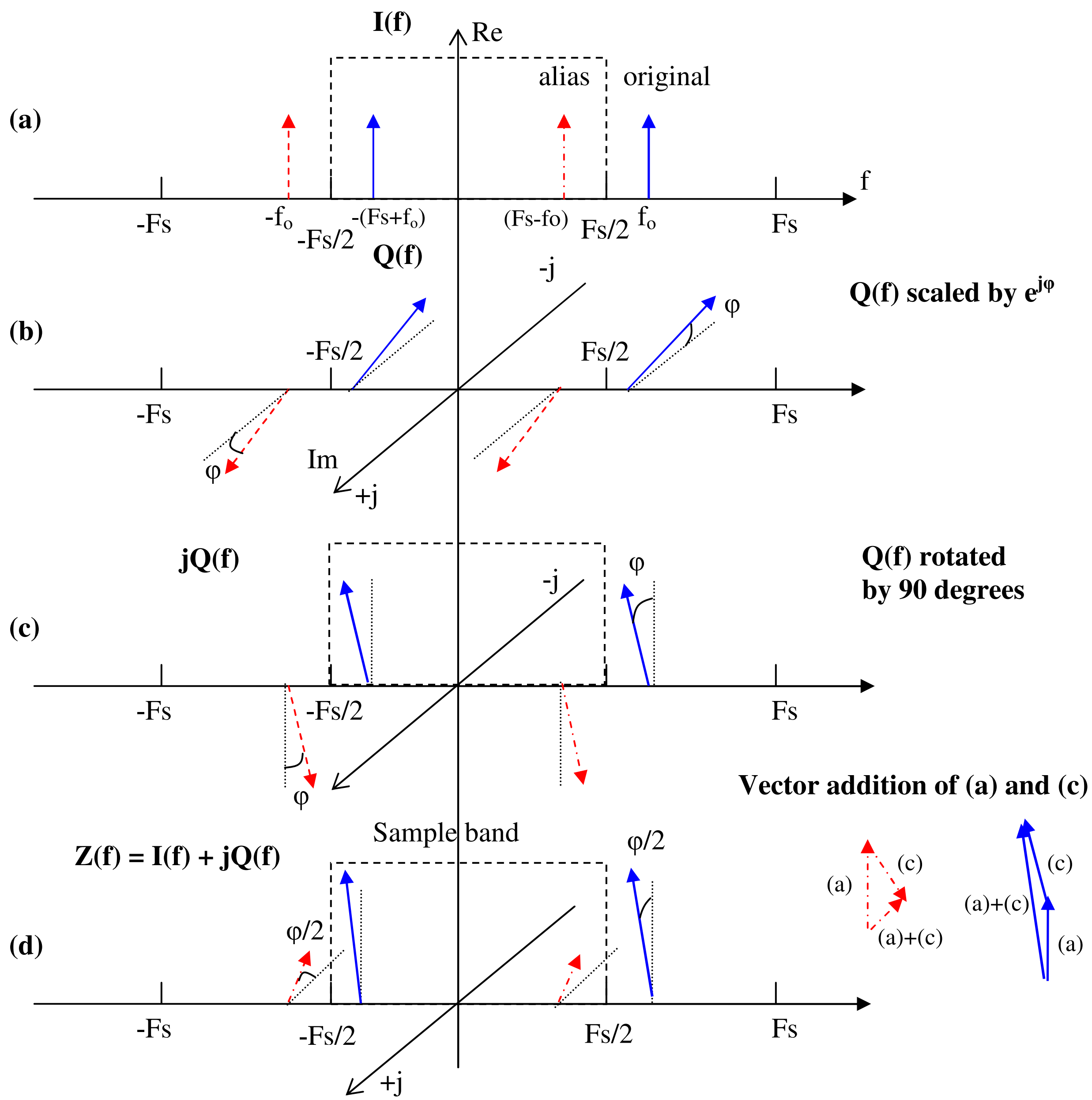


Fig 4. – Spectrum of complex-valued signal with a phase imbalance angle = φ

As shown before, we create the complex-valued signal by adding the I signal and the "rotated" Q signal. With a phase imbalance on Q, one can see that the quadrature signal created by adding Fig 4(a) and (c) does not perfectly cancel the negative frequency (alias) images. In other words, the signal is no longer analytic because of the imbalance.

We can also see that the positive freq term is reduced in magnitude, and the negative frequency exponential term now contributes to the overall spectrum.

The result is shown in Fig 4(d). If we attempt to combine both real-valued signals into a complex-valued signal Z(f), we will not get perfect alias (image) cancelation. The negative frequency term in the sample band will be seen as distortion in our resultant quadrature signal. This violates our assumption that the complex-valued signal will be analytic. And because it is no longer analytic, we will not be able to resolve (without ambiguity) frequencies in the [Fs/2, Fs] band (even though the resultant signal is complex-valued).

Amplitude imbalance

If the quadrature input signal also has some amplitude error between the I and Q channels, this also couples with the phase imbalance. Amplitude imbalance also creates disortion in the form of a negative frequency image.

Let's now write a quadrature signal with a phase offset of φ radians, the I channel with an amplitude of α, and the Q channel with an amplitude of β.

$$A(t) = \alpha\cos(wt) + j\beta\sin(wt+\phi) \quad (1.8)$$

Eq. (1.8) can also be written in complex-form as:

$$A(t) = \frac{1}{2}\left[e^{jwt}\cdot(\alpha+\beta e^{j\phi}) + e^{-jwt}\cdot(\alpha-\beta e^{-j\phi})\right] \quad (1.9)$$

As before in the phase imbalance case, the signal is no longer analytic and the negative frequency image (that is seen as a distortion in the output) has a magnitude of:

$$\left|\frac{1}{2}(\alpha-\beta e^{-j\phi})\right| \quad (1.10)$$

Thus we can see the amplitude imbalance terms also play a role in the magnitude of the negative frequency image in the output complex-valued signal. The amplitude and phase imbalances are indeed coupled and directly affect the strength of the negative frequency image.

Of course, if we make $\alpha=\beta$, then Eq. (1.9) is equal to Eq. (1.7) with a simple scaling factor. Also note that for $\alpha=\beta$ and $\varphi=0$, Eq. (1.9) still reduces to the well-known complex exponential.

Digital Quadrature Phase Balancing

To address phase balancing, we go back to our mixdown example in Fig. 2, where the input signal is a complex sinusoid being mixed down to an intermediate frequency (IF).

To rectify any phase and amplitude imbalance in the input signal, it turns out that we can correct these in the digital domain before the complex mix to baseband. Thus, we can take an ill-formed analog (or digital) quadrature signal and "fix" it with DSP techniques!

Let's first consider the phase balancing task by itself. Here we will assume there is no amplitude imbalance (we will see later it can be added in the path without loss of generality).

The first task is to estimate the phase imbalance angle φ. Once we have estimated this parameter, we can correct for it digitally. There are several methods on how to do this phase correction – two such methods are proposed. Estimating the phase imbalance angle will be presented after the methods of how to actually "fix" the imbalance.

<u>Method 1:</u>

The first method involves using a fractional-delay filter to delay the Q channel. In this setup, the phase imbalance φ is used to determine how many samples we must delay the Q channel with respect to the I channel to maintain a 90-degree phase relationship.

With this method we would be delaying the Q channel by "d" samples, where d is not constrained to be an integer:

$$Q_{corr}(n) = Q(n-d), \text{ where } d = \varphi/2\pi fT$$

The fractional-delay filter will have a total delay D = d + N. The FIR delay "N" is the inherent delay in the filter itself (related to the number of filter taps), and this delay is added to the desired fractional delay term to create "total delay" D.

Here is a block diagram:

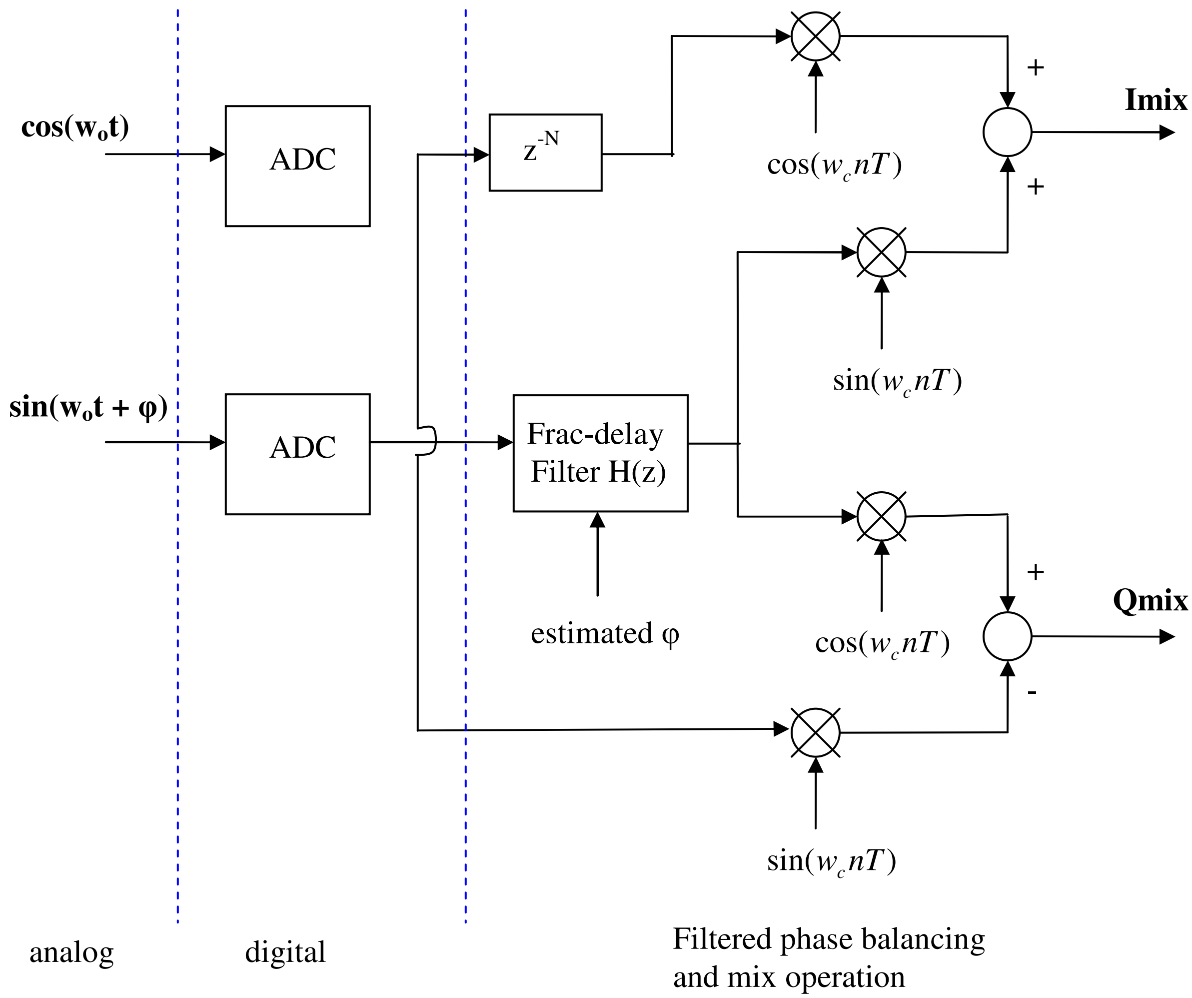


Fig. 5 – Fractional-delay FIR filter for phase correction

The fractional-delay filter shown in Fig.5 would be generated once the system has estimated the phase imbalance term. The delay z^{-N} in the I signal path is strictly there to time align the 2 signals. Since the Q term will be inherently delayed due to the FIR filter in it's path, we must also delay the I term by the integer number of samples due to the filter delay (N). The composite delay leftover between I and Q will be the fractional delay "d" we wanted.

A low-order FIR filter could be generated using Lagrange interpolation coeffs for the fractional-delay filter [4]:

$$h[n] = \prod_{\substack{k=0 \\ k \neq n}}^{N} \frac{D-k}{n-k}, n = 0,1,2,..., N$$

This filter has a great approximation for low frequencies and the filter coefficients are easy to compute. But there is a phase delay vs. magnitude response tradeoff with this filter structure. Other filters can be used here to meet the necessary specs – including a least squares filter or a polyphase fractional-delay structure.

Method 2:

The second method of phase correction comes from the serendipitous application of the basic trig identity:

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

In which if we apply to our phase imbalance signal on the Q channel:

$$\sin(wt+\phi) = \sin(wt)\cos(\phi) + \cos(wt)\sin(\phi)$$

Now we can solve for sin(wt), which is the desired output we wish to generate given our phase-imbalanced input signal:

$$\sin(wt) = \frac{\sin(wt+\phi) - \cos(wt)\sin(\phi)}{\cos(\phi)} \quad (1.11)$$

And referring to Eq. 1.1b, since cos(wt) = I channel and sin(wt+φ) = Q channel, we can use these two inputs to generate a corrected Q signal via:

$$\sin(wt) = Q_{corr} = \frac{Q - I \cdot \sin(\phi)}{\cos(\phi)} \quad (1.12)$$

Where the phase imbalance angle φ would again be estimated using digital samples.

Since cos(φ) is always less than or equal to one, that means the denominator term in (1.12) will be greater than one, which is not easily implemented in fixed-pt hardware. Thus if we are implementing this on a fixed-pt FPGA or DSP, we must rewrite this equation to be implemented in fixed-pt (integer) math. One equation to use is:

$$Q_{corr} = (Q - I \cdot sf) + (Q - I \cdot sf)\cdot(cf) \qquad (1.12b)$$

Where: sf = sin(φ) and cf = (1/cos(φ)-1), if 0<φ<60 deg

This “direct method” of phase correction calls for 2 multiplies and 2 adds, and can be readily implemented inside an FPGA or DSP. Also note we can turn off this correction by setting the factors sf = 0 and cf = 0.

Next is a block diagram of the direct method:

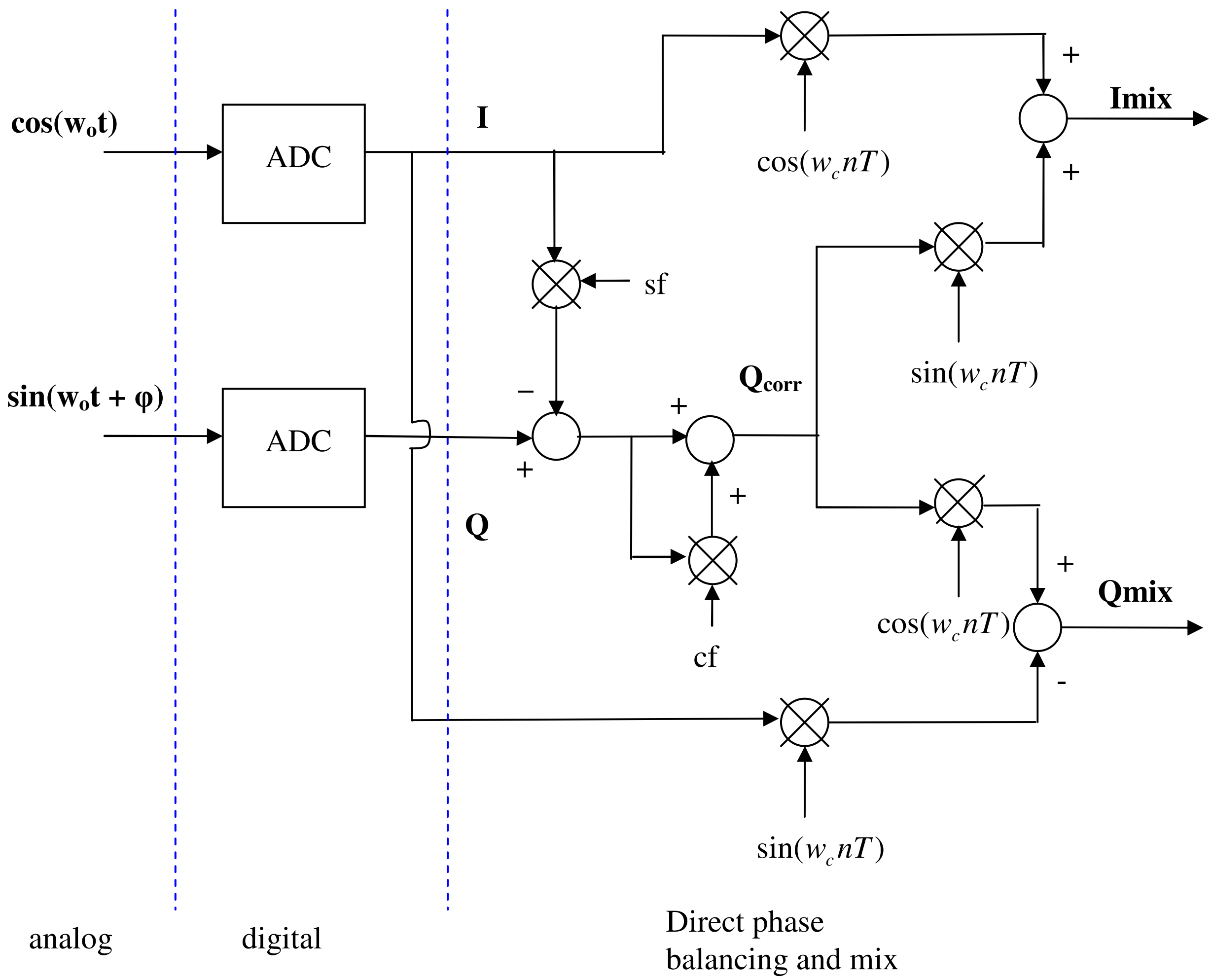


Fig. 6 – Direct phase correction method

For the direct-phase correction method, we have to specify the proper number of bits so our fixed-pt multiply-accumulate gives sufficient correction of the phase imbalance. A typical coefficient length would be 14-24 bits, depending on the necessary accuracy and the resources available.

One other method of doing phase imbalance correction combined with a complex mixing operation was considered. This method involved doing all of the phase imbalance correction and mixing in one operation - by adding phase correction terms to the mixing tables to remove any phase imbalance. This approached seemed elegant but was proved to be intractable [6].

Phase Imbalance Estimation

We already have a method for correcting the phase of the input quadrature signal, but we first must estimate the actual phase imbalance angle φ. This estimation is more readily done in a DSP or other processor where multiple time-series digital samples can be analyzed.

One of the simplest methods is to treat the I and Q signals as vectors in N^{th}-dimensional space. Since each signal can be defined as an element in *Hilbert space*, then we can use the definition on the inner (dot) product to estimate the angle between the two signals.

Hence,

$$\langle a,b\rangle = |a|\cdot|b|\cdot\cos(\theta) \qquad (1.13)$$

Where the modulus is defined as: $|a| = sqrt(\langle a,a\rangle)$

To find the phase angle between the incoming I and Q signals, simply use the dot product definition over the number of samples collected.

Let us define the inner product in Hilbert space (where I and Q can be complex),

$$\langle I,Q\rangle = \sum_{k=0}^{N-1} I_k Q_k^*\,,\ \text{and N = number of samples collected}$$

And,

$$|I| = sqrt\left(\sum_{k=0}^{N-1} I_k^2\right),\ |Q| = sqrt\left(\sum_{k=0}^{N-1} Q_k^2\right)$$

Solving for the phase angle in Eq. (1.13),

$$\theta = \cos^{-1}\left(\frac{\langle I,Q\rangle}{|I|\cdot|Q|}\right) \qquad (1.14)$$

$$\therefore \phi = \frac{\pi}{2} - \theta$$

These operations are easily done on incoming complex-valued data samples. Also note that this is not a temporal operation, but a *vector operation*. Hence we do not need contiguous data samples for this method to work, we just need N samples of the signal in order to estimate the phase angle (fragmented or contiguous).

Once we have determined the phase imbalance angle φ, we can set the sin() and cos() factors to correct the digital Q signal. Once this is done, the incoming analog signal will now be forced to have a 90-degree separation (analytic), and the resulting complex mix will be free from negative frequency images due to phase imbalances.

MATLAB Simulation

Next is a matlab simulation of an imbalanced input signal using floating-pt math. The Q channel has been arbitrarily skewed by 10 degrees. The inner product approach was used to estimate the phase angle between the I and Q channels.

```
>> x=cos(2*pi*1e6/128e6*[1:200000]);
>> y = sin(2*pi*1e6/128e6*[1:200000] + 10*pi/180);
```

The estimator ran over 200,000 samples and returned the following phase angle estimate:

```
>> 180/pi*acos(dot(x,y)/(sqrt(sum(x.^2))*sqrt(sum(y.^2))))

ans =

  79.999999999989683
```

Then the above direct method was used to correct for the phase imbalance.

```
>> corr = pi/2-ans*pi/180

corr =

   0.174532925199613

>> sf = sin(corr)

sf =

   0.173648177667108

>> cf = 1/cos(corr)

cf =

   1.015426611885777
```

Now compute the new Qcorr channel using the phase estimate:

```
>> y2 = (y-x*sf)*cf;
```

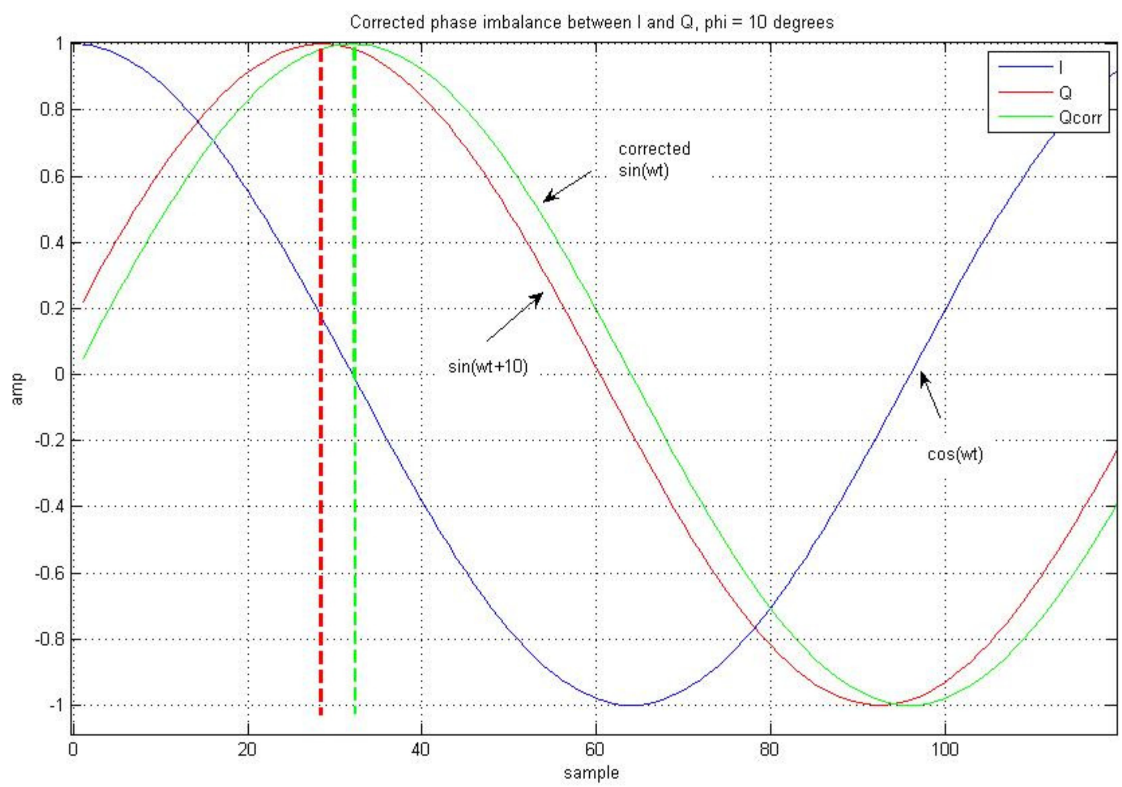
If we check the new angle between the I and Qcorr signals, the corrected phase angle is indeed correct:

```
>> 180/pi*acos(dot(x,y2)/(sqrt(sum(x.^2))*sqrt(sum(y2.^2))))

ans =

  90.000000000005485
```

Below is a plot of the resulting output quadrature signal, showing that we have successfully corrected the Q channel back to a 90-degree offset with respect to the I channel.



Next we can simulate a fixed-pt implementation using 16-bit scalars and input signals.

Here is the result of that simulation. First quantize the input signals to 16 bits:

```
>> xq=fxquant(x,16,'round','sat');
>> yq=fxquant(y,16,'round','sat');
```

```
>> 180/pi*acos(dot(xq,yq)/(sqrt(sum(xq.^2))*sqrt(sum(yq.^2))))

ans =

  79.999843652984907
```

The estimated phase angle is accurate to 1e-4 degrees using 16-bit input samples in the phase angle estimator.

```
>> corr = pi/2-ans*pi/180

corr =

   0.174535653969622

>> sf = sin(corr)

sf =

   0.173650864980322

>> sf = fxquant(sf,16,'round','sat')

sf =

   0.173645019531250

>> cf = 1/cos(corr)

cf =

   1.015427100468173
```

Here we take $1/\cos(\varphi)-1$ as outlined before for a fixed-pt implementation:

```
>> cfq = fxquant(cf-1,16,'round','sat')

cfq =

   0.015441894531250
```

Here is the correction done in the FPGA using 16-bit fixed pt multiplies, utilizing Eq. (1.4b):

```
>> y2q = fxquant((yq-fxquant(xq*sf,16,'round','sat'))*(1+cfq),16,'round','sat');
```

Checking our result gives us the corrected phase angle between I and Qcorr:

```
>> 180/pi*acos(dot(xq,y2q)/(sqrt(sum(xq.^2))*sqrt(sum(y2q.^2))))

ans =

  89.999550699810129
```

So using the proposed fixed-pt implementation with 16-bit samples and multiplies, we still get a phase angle correction accurate down to 1e-4 degrees.

Digital Quadrature Amplitude Balancing

Up to this point we have brushed aside the amplitude imbalance issue. The reason for this is that our phase balancing derivation assumed equal amplitudes. So if we simply scale the I and Q channels to have equal amplitudes and pass the resulting signals to the phase correction block, we will have solved the coupled problem of phase and amplitude balancing.

Amplitude correction is easily implemented in a similar fashion by estimating the envelope of the I and Q signals and correcting them with scalar multiplies before the phase correction block. We can also push the relative amplitude difference into the Q term to rid of one extra multiply.

Figure 7 shows one such block diagram that does amplitude correction first, then phase correction.

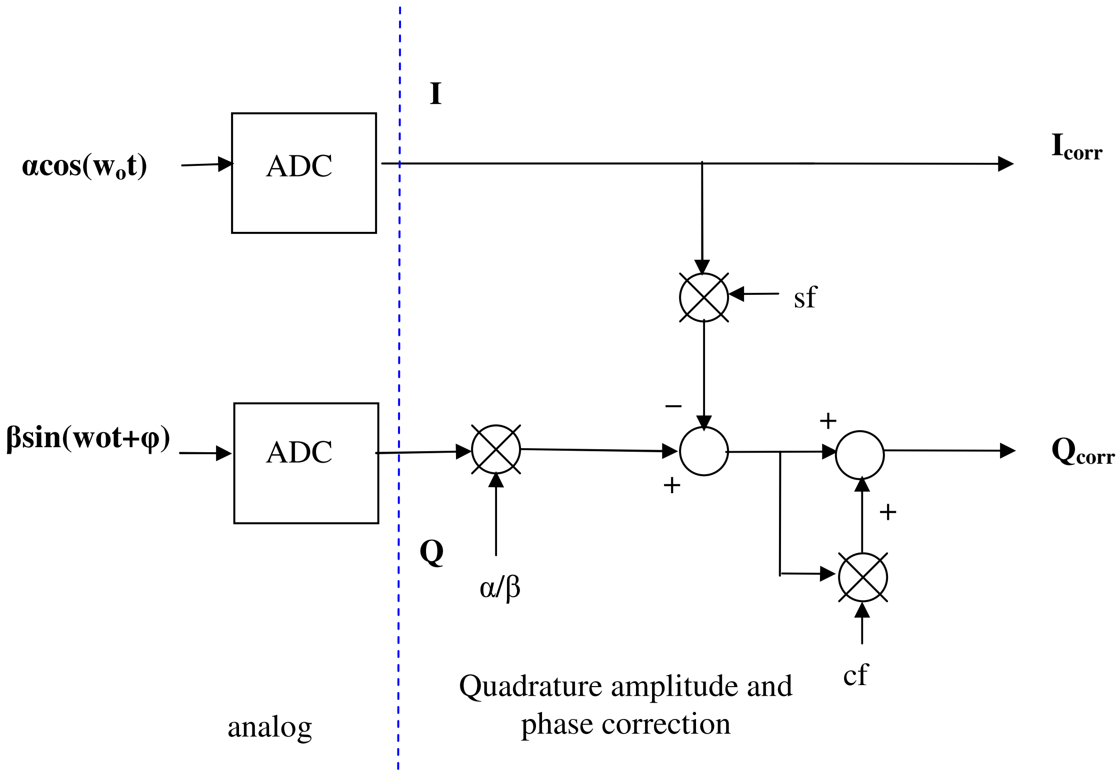


Fig. 7 – Amplitude and phase correction for imbalanced quadrature input

In determining the amplitudes α and β for the I and Q signals, one can use a similar approach to the imbalance phase angle estimation. This time we can look at a contiguous time series of "N" digital samples. If we assume again we are expecting sinusoidal input, we can use Parseval's theorem to compute amplitude from power. Recall Parseval's theorem states that power computed in the frequency domain and time domain are equal.

As such, we can compute power (given a zero-mean signal) via:

$$P_o = \frac{1}{N}\sum_{n=0}^{N-1}(x[n])^2 \quad \text{“Parseval's theorem”}$$

$$P_o = \frac{A^2}{2} \quad \text{“power in a sinusoid”} \quad (1.15)$$

First compute the power P_o over N pts, then solve for the amplitude "A" of the sinusoid using Eq. (1.15).

NOTE: The above approach guarantees that you will get the "true" amplitude. Consider if you simply tried to look at the max and min samples of your signal to determine the amplitude. Then you might not ever see the true max or min amplitude value. In fact, there are certain frequency relationships between input and sample frequency where you will never see the true max and min amplitudes in the digital samples! The above approach mitigates this issue altogether.

Conclusion

This article focused on quadrature signaling and the definition of complex-valued signals with respect to real-world inputs. The effects of phase and amplitude imbalances in quadrature signals were explored, and methods for compensating these imbalances were discussed.

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APPENDIX [6]

One other method researched for direct-phase correction with complex mixing was to fold the phase correction terms into the complex mix operation. This would make sense since the multiply operations by the sine and cos mixer tables could be used to shift the Q input signal and constrain it to be analytic while doing the mixdown. As promising as this solution appears, the below proof dashes all hopes of it ever working.

We will prove that the above hypothesis is impossible via ***reductio ad absurdum***.

Let the input signal w/ a phase imbalance φ be defined as in Eq. (1.7):

$$A_\phi(t) = \frac{1}{2}\left[e^{jw_c t}\cdot(1+e^{j\phi}) + e^{-jw_c t}\cdot(1-e^{-j\phi})\right]$$

And let us define our complex mixer to have 2 phase correction terms, α and β, that we can optimize to cancel the input phase imbalance. Hence, instead of mixing by a complex exponential $m(t) = e^{-jwt}$, we would now mix the input down using:

$$m(t) = \cos(w_o t + \alpha) - j\sin(w_o t + \beta)$$

Or,

$$m(t) = \frac{1}{2}\left[e^{jw_o t}\cdot(e^{j\alpha} - e^{j\beta}) + e^{-jw_o t}\cdot(e^{-j\alpha} + e^{-j\beta})\right] \quad (1.15)$$

Also note that m(t) is no longer analytic but is symmetric and complex-valued, just like $A_\varphi(t)$.

The question remains: Can we multiply 2 non-analytic complex-valued signals and create an analytic result? What we want from the mix is the following:

$$A_\phi(t)m(t) = e^{j(w_c - w_o)t} \quad \text{(desired result)} \quad (1.16)$$

If we multiply both signals in Eq. (1.7) and (1.15) (perform the mix), we get the following result.

$$\begin{aligned} A_\phi(t)m(t) = \frac{1}{4}[&e^{j(w_c+w_o)t}\cdot(e^{j\alpha} - e^{j\beta})(1+e^{j\phi}) \\ &+ e^{j(w_c-w_o)t}\cdot(e^{-j\alpha} + e^{-j\beta})(1+e^{j\phi}) \\ &+ e^{-j(w_c-w_o)t}\cdot(e^{j\alpha} - e^{j\beta})(1-e^{j\phi}) \\ &+ e^{-j(w_c+w_o)t}\cdot(e^{-j\alpha} + e^{-j\beta})(1-e^{-j\phi})] \end{aligned} \quad (1.8)$$

From this mix operation, we can see we only want to keep the 2nd term, which is the desired analytic complex exponential in Eq. (1.1b), but scaled by a complex value. All other terms are images we want to suppress if we want a true analytic output signal.

We can write the following set of constraints to achieve the desired signal as in Eq (1.1b):

$$(1)\quad e^{j\alpha}(1+e^{j\phi}) - e^{j\beta}(1+e^{j\phi}) = 0$$
$$(2)\quad e^{-j\alpha}(1+e^{j\phi}) + e^{-j\beta}(1+e^{j\phi}) = 4e^{j\lambda}$$
$$(3)\quad e^{j\alpha}(1-e^{-j\phi}) - e^{j\beta}(1-e^{-j\phi}) = 0$$
$$(4)\quad e^{-j\alpha}(1-e^{-j\phi}) + e^{-j\beta}(1-e^{-j\phi}) = 0$$

Uniquely solving these 4 constraints will create an analytic signal and rid of the offending images due to the phase imbalance.

One can see that constraints (1) and (3) require that:

$$e^{j\alpha} = e^{j\beta}\text{, or } \alpha = \beta + 2n\pi$$

Whereas constraint (4) requires:

$$e^{-j\alpha} = -e^{-j\beta}$$

Plugging in for α:

$$e^{-j(\beta+2n\pi)} = -e^{-j\beta}$$
$$1 = -1$$

Which is a contradiction! Therefore there does not exist any (α, β) pair to solve the problem.

In short, we cannot cancel out all the images in the mixing operation while simultaneously creating an analytic result. We must fix the phase imbalance before mixing. The above has proved that two non-analytic complex-valued signals cannot be mixed together with the side-effect of generating a purely analytic result.